Statistical Characteristics of Reynolds Stresses in a Turbulent Boundary Layer

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Measurements are presented of autocorrelations and spectra of the Reynolds normal and shear stress fluctuations in a turbulent boundary layer. The measurements in the fully turbulent part of the layer are in good agreement with calculations which assume a Gaussian probability density of the individual velocity fluctuations and a joint Gaussian probability density for these fluctuations. The measured spectral densities scale reasonably well with freestream velocity and boundary-layer thickness over the whole frequency range, throughout the boundary layer.

I. Introduction

THE statistics of shear stress fluctuations in a turbulent boundary layer were first considered by Corrsin.[1] Probability densities of the Reynolds shear stress fluctuations have been measured in a turbulent boundary layer under zero pressure gradient conditions by Gupta and Kaplan,[2] Antonia and Atkinson,[3] Hedley and Keffer,[4] and Lu and Willmarth.[5] These measured densities have been compared with the calculated density obtained by assuming a normal joint probability density $p\ (u,v)$ for the longitudinal u and normal v fluctuations. Comparison between calculated and measured probability densities has been found to be reasonable in the inner region of the boundary layer, not too close to the wall. Hedley and Keffer[4] found that both in the inner and outer region of the layer the point maxima of the $p(u,v)$ contours occurred in the lower right-hand quadrant $(u>0,v<0)$ of the uv plane but the contours are skewed towards the upper left-hand quadrant $(u<0,v>0)$. Improved agreement between the measured and calculated third- and fourth-order moments of uv was obtained by Antonia and Atkinson[3] by making allowance for small departures from Gaussianity of $p(u,v)$ in the turbulent part only of the flow.

In this paper, we assume that the densities $p(u)$, $p(v)$, and $p(u,v)$ are Gaussian to calculate the autocorrelation functions of the instantaneous stresses u^2, v^2, and uv. The second-order spectra associated with these stresses then are obtained via Fourier transforms of certain functions of the correlation functions. Second and higher order spectra of u were obtained by Champagne et al.[6] in a two-dimensional mixing layer as an exercise to determine if their instruments had sufficient high-frequency response to measure all spectral contributions to the higher order moments. Van Atta and Wyngaard[7] calculated high-order spectra of u, based on a Gaussian probability density for the u fluctuations, and found them to be in good agreement with the spectra measured in the atmosphere and with those of Champagne et al.[6] The work in this paper represents an extension of the method of Van Atta and Wyngaard[7] to the calculation of the spectra of v^2 and uv. The calculated (Gaussian) spectra of u^2, v^2, and uv are compared with measured spectra obtained directly from digital records of u and v.

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II. Experimental Conditions

Measurements were made in a smooth wall turbulent boundary layer with a zero pressure gradient. The boundary layer was developed on the floor of the working section of a wind tunnel which was of the open type, with a 9:1 contraction feeding a 38.1-cm-wide by 22.8-cm-high working section. The working section was 4.88 m long and had a variable roof geometry to allow the pressure gradient to be set to zero. A 3.2-mm-diam. circular rod was used to trip the boundary layer at the start of the working section. The measuring station was 2.44 m downstream of the trip. At this station, the 99.5% boundary-layer thickness δ was approximately 5 cm and the momentum Reynolds number R_θ was equal to 2190, when the freestream velocity U_1 was equal to 5.5 m/s $^{-1}$.

The velocity fluctuations u (streamwise) and v (normal) were obtained with a miniature DISA X probe with a 2.35-mm diam. and 8.3-cm-long stem situated parallel to the wall. The 5-mm-diam. Platinum coated tungsten wires were welded to the ends of prongs, approximately 7 mm upstream of the probe stem. The X probe was operated by two channels of constant temperature anemometers. The signals from the X wire first were passed through 1 kHz cutoff low-pass filters, then sampled at a frequency of 3 kHz (per channel) before being recorded on digital tape. This tape was processed on an English Electric KD9 computer. Records of u and v first were formed from the digitized anemometer voltages and spectra of u, v, uv, and of the squares of these values were obtained by application of a fast Fourier transform (*FFT*) algorithm. The Fourier transform was applied either to a block of 512 characters with the required spectral densities formed by subsequently averaging over 64 consecutive blocks or else applied directly to a block of 2^{15} characters, i.e., a record of 10.92-sec duration. The autocorrelations and cross correlations of $u^m v^n$ (m, $n=0$, 1, 2) were obtained by doing an inverse *FFT* on the relevant spectral densities. Further information on the hot-wire instrumentation, digital data acquisition system or the *FFT* algorithm used may be found in Refs. 8 and 9.

III. Autocorrelations of Reynolds Stresses

The correlation coefficients $R_{u^2-\langle u^2\rangle}$ and $R_{v^2-\langle v^2\rangle}$ of the fluctuations $(u^2(t)-\langle u^2\rangle)$ and $(v^2(t)-\langle v^2(t)\rangle)$, respectively, are defined here by

$$R_{u2-<u2>} = \frac{<(u^2(t)-<u^2(t)>)(u^2(t+\tau)-<u^2(t+\tau)>)>}{<u^4>-<u^2>^2} \quad (1)$$

and

$$R_{v2-<v2>} = \frac{<(v^2(t)-<v^2(t)>)(v^2(t+\tau)-<v^2(t+\tau)>)>}{<v^4>-<v^2>^2} \quad (2)$$

It follows that

$$R_{u2}-1=R_{u2-<u2>}(F_u-1) \quad (3)$$

and

$$R_{v2}-1=R_{v2-<v2>}(F_v-1) \quad (4)$$

where

$$R_{u2}=<u^2(t)u^2(t+\tau)>/<u^2>^2$$

$$R_{v2}=<v^2(t)v^2(t+\tau)>/<v^2>^2$$

and F_u, F_v are the flatness factors of u and v, respectively, (u and v have zero mean). For a jointly Gaussian probability density function of $u(t)$ and $u(t+\tau)$

$$R_{u2}=1+2(R_u)^2 \quad (5)$$

where $R_u=<u(t)u(t+\tau)>/<u^2>$. The analogous relation for R_{v2} is

$$R_{v2}=1+2(R_v)^2 \quad (6)$$

and

$$\left[\frac{R_{u2-<u2>}(\tau)}{R_u^2(\tau)}\right]=\frac{2}{F_u-1}, \quad \frac{R_{v2-<v2>}(\tau)}{R_v^2(\tau)}=\frac{2}{F_v-1} \quad (7)$$

The experimental values of $2/(F_u-1)$ and $2/(F_v-1)$ are compared in Fig. 1 with the Gaussian value of unity. It is clear that the assumption of Gaussianity is only approximately valid in the inner region of the flow $0.1<y/\delta<0.4$, excluding the region very close to the wall. The non-Gaussianity of the statistics of u and of the (not easily measured) v fluctuations in this latter region is well in evidence in the literature (Ref. 10).

The correlations R_u, $R_{u2-<u2>}$ and R_v, $R_{v2-<v2>}$ are presented in Figs. 2 and 3, respectively, for $y/\delta=0.08$. The Gaussian values of $R_{u2-<u2>}$ and $R_{v2-<v2>}$ obtained from (7) and normalized by the respective values at $\tau=0$ also are shown on these plots. For short time lags, the agreement between the Gaussian values and the measured curve is good. Poorer agreement is obtained at large time lags, especially in the case of v, probably as a result of the large-scale structure of v being inhibited by the presence of the wall.

If, in addition to Gaussian probability density functions of u and v, we assume that the joint probability density of u and v is Gaussian, then

$$\begin{aligned}<u(t)v(t)u(t+\tau)v(t+\tau)> &= <u(t)v(t)><u(t+\tau)v(t+\tau)> \\ &+<u(t)u(t+\tau)><v(t)v(t+\tau)> \\ &+<u(t)v(t+\tau)><v(t)u(t+\tau)>\end{aligned} \quad (8)$$

Assuming stationarity, and dividing (8) by the product $<u^2><v^2>$, we find

$$R_{uv-<uv>}(\tau)=R_u(\tau)R_v(\tau)+R_{u,v}(\tau)R_{u,v}(-\tau) \quad (9)$$

where $R_{uv-<uv>}(\tau)$ is the autocorrelation coefficient of $uv-<uv>$, i.e.

$$R_{uv-<uv>}(\tau)=\frac{<u(t)v(t)u(t+\tau)v(t+\tau)>-<uv>^2}{<u^2><v^2>} \quad (10)$$

and the cross correlation coefficient $R_{u,v}(\tau)$ is

$$R_{u,v}(\tau)=\frac{<u(t)v(t+\tau)>}{<u^2>^{1/2}<v^2>^{1/2}} \quad (11)$$

with $R_{u,v}(-\tau)$ being equal to $R_{v,u}(\tau)$.

At $\tau=0$, (9) reduces to

$$R_{uv-<uv>}(0)=1+R_{u,v}^2(0) \quad (12)$$

Experimental values of $R_{uv-<uv>}(0)$, shown in Fig. 4 compare favorably in the range $0.1<y/\delta<0.4$, with the values obtained from (12) using the experimental values of $R_{u,v}(0)$. The difference between the measured values of $R_{uv-<uv>}$ and those calculated from (12) becomes large in the intermittent part of the flow.

Note that (12) is identical with expressions for $<u^2v^2>$ derived in Lumley[11] and Antonia and Atkinson[3] from the characteristic function of the jointly Gaussian density of u and v. Values of $<u^2v^2>^{1/2}/\tau_w$ presented by Gupta and Kaplan[2] and Antonia[12] in the inner region of a turbulent boundary layer are in reasonable agreement with the atmospheric value of Wyngaard and Izumi[13] indicating no significant Reynolds number dependence of this ratio.

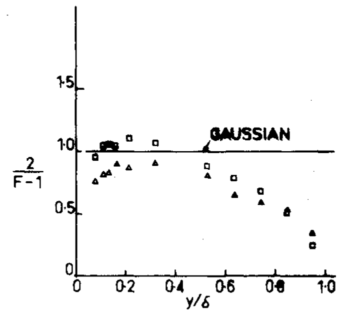

Fig. 1 Departure from Gaussianity of correlation coefficients $R_{u2-<u2>}$ and $R_{v2-<v2>}$ at zero time delay as a function of position in the layer; □, $2/(F_u-1)$; Δ, $2/(F_v-1)$ [Eq. (7)].

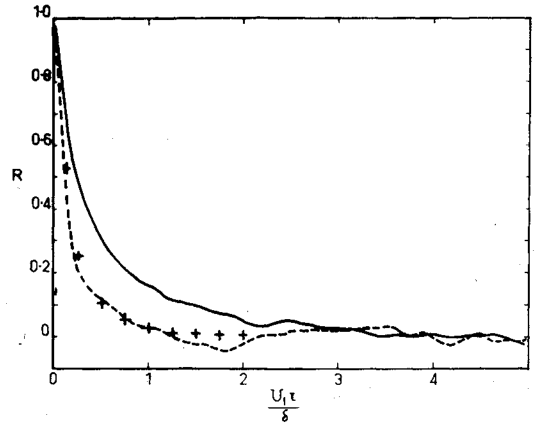

Fig. 2 Autocorrelations of u and $(u^2-\langle u^2\rangle)$ fluctuations at $y/\delta=0.08$; – – –, R_u measured; --, $R_{u2-\langle u2\rangle}$ measured; +, $R_{u2-\langle u2\rangle}$ calculated using Gaussian assumption [Eq. (5)].

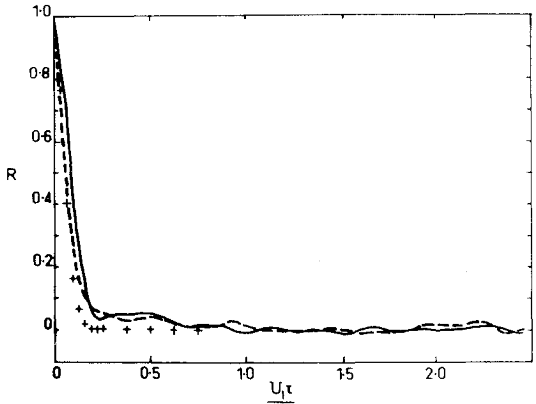

Fig. 3 Autocorrelations of v and $(v^2 - \langle v^2 \rangle)$ fluctuations at $y/\delta = 0.08$; ——, R_v measured; – –, $R_{v2-\langle v2 \rangle}$ measured; +, $R_{v2-\langle v2 \rangle}$ calculated using Gaussian assumption [Eq. (6)].

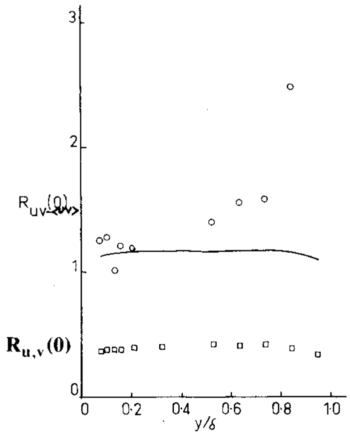

Fig. 4 Autocorrelation coefficient of uv product at zero time delay; ○, $R_{uv-\langle uv \rangle}$ *(0)* measured; ——, $R_{uv-\langle uv \rangle}$ *(0)* calculated using Gaussian assumption [Eq. (9)]; □, $R_{u,v}$ *(0)* measured.

When the third- and fourth-order moments of uv are considered, the small departure from Gaussianity in the inner region of the layer of the joint probability density of u and v must be taken into account to calculate the skewness and flatness factor of uv correctly. Antonia and Atkinson[3] obtained satisfactory results with a cumulant-discard approach and the exclusion of correlations between u and v higher than the fourth order. The joint probability density function which results from this approach is equivalent to the Gram-Charlier distribution used by Frenkiel and Klebanoff[14] to calculate higher order correlations $\langle u^m(t)u^n(t+\tau) \rangle$ in the inner region of a turbulent boundary layer. Here, we are interested only in the fourth-order correlations R_{u2}, R_{v2} and $R_{uv-\langle uv \rangle}$, and we make no attempt to account for the non-Gaussianity of $p[u_I(t), u_2(t+\tau)]$, where u_I and u_2 stand for either u or v, in the analysis.

The measured auto correlation $R_{uv-\langle uv \rangle}$, shown in Fig. 5, is in good agreement at small τ with the Gaussian result obtained from (9). Both the Gaussian and measured values have been normalized by their respective values at $\tau = 0$. Distributions of the measured cross-correlation coefficients used in (9) also are shown in Fig. 5 normalized by the respective values at the origin. $R_{u,v}(\tau)$ is clearly not an even function of τ. It has a maximum value of 0.37 at $U_I\tau/\delta \simeq 0.031$.

Integral time scales corresponding to the fluctuations u, v, uv, u^2, v^2 have been obtained by integrating the normalized correlation curves shown in Figs. 2, 3, and 5. The integration arbitrarily was carried out to the first zero crossing τ_I and these time scales then were converted to length scales

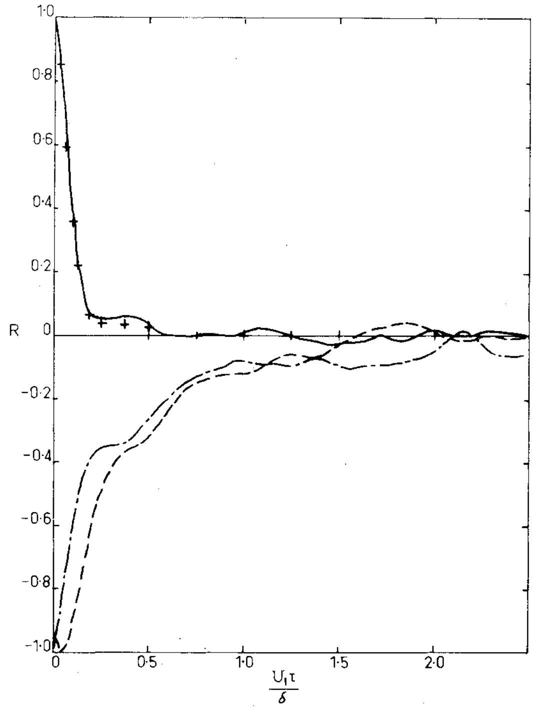

Fig. 5 Autocorrelation coefficient of uv product and cross correlation between u and v at $y/\delta = 0.08$; —— $R_{uv-uv}(\tau)$ measured; +, $R_{uv-\langle uv \rangle}(\tau)$ calculated using Gaussian assumption [Eq. (9)]; – –, $R_{u,v}(\tau)/[R_{v,u}(\tau)]_{\max}$.

with the use of Taylor's hypothesis. The length scale of a quantity x is defined here as

$$L_x = U \int_0^{\tau_I} \frac{R_x(\tau)}{R_x(0)} d\tau$$

Values of L_u/δ, shown in Fig. 6, are in reasonable agreement with data, available in the literature, taken for boundary layers under conditions similar to those for the present experiment. The magnitude of L_{u2} is roughly one half that of L_u whereas the values of L_v, L_{v2} are approximately the same, emphasizing the relative lack of low-frequency energy in the v spectrum. Values of L_{uv} are the smallest encountered in Fig. 6. This result is not unreasonable since Eq. (9) involves the products R_u R_v and $R_{u,v}$ $R_{v,u}$. Autocorrelations of uv fluctuations have been measured well within the viscous sublayer by Gupta[15] in the boundary layer and by Wallace et al.[16] in a fully developed channel flow. The latter authors also have obtained correlation contributions to the uv signal from each of the four quadrants in the (u,v) plane.

IV. Spectra of Reynolds Stresses

Measured power spectra ϕ_{uv} of the fluctuations of uv about the mean $\langle uv \rangle$ are shown in Fig. 7 at several positions in the boundary layer. The normalizing scales are U_I and δ and the spectral densities are divided by $\langle (uv)^2 \rangle - \langle uv \rangle^2$, so that the area under each spectrum is equal to unity, i.e.

$$\int_0^\infty \phi_{uv}\left(\frac{\omega\delta}{U_I}\right) d\left(\frac{\omega\delta}{U_I}\right) = \langle (uv)^2 \rangle - \langle uv \rangle^2$$

Reasonable similarity is observed for all spectra shown in Fig. 7 over the whole frequency range. The spectrum at $y/\delta = 0.08$ also was calculated from the "Gaussian" $R_{uv}(\tau)$ curve shown in Fig. 5, using the Fourier transform relation

$$\phi_{uv}(\omega) = \langle u^2 \rangle \langle v^2 \rangle \int_0^\infty R_{uv}(\tau) e^{-i\omega\tau} d\tau \tag{13}$$

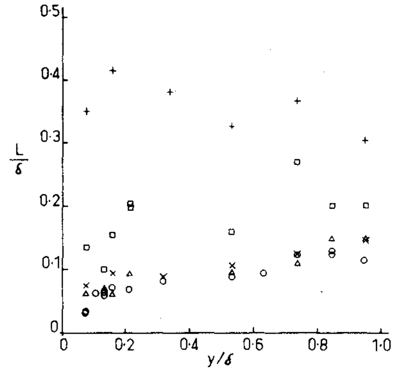

Fig. 6 Measured integral length scales across the boundary layer; +, $x \equiv u$; ×, v; □, $u^2 - \langle u^2 \rangle$; Δ, $v^2 - \langle v^2 \rangle$; ♂, $uv - \langle uv \rangle$.

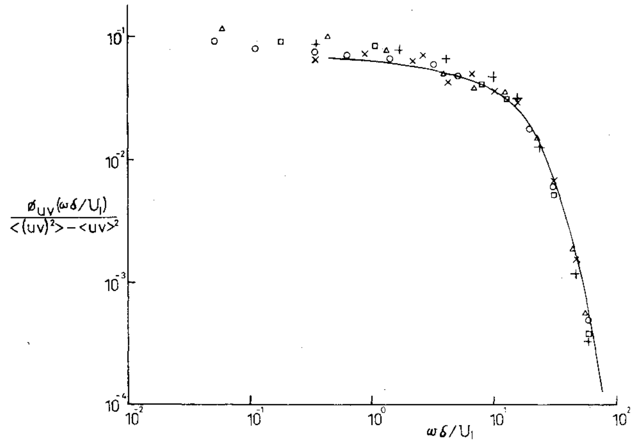

Fig. 7 Spectra of Reynolds shear stress $uv - \langle uv \rangle$ fluctuations at several positions across the layer. Measurements: ×, $y/\delta = 0.08$; Δ, 0.105; ○, 0.21; +, 0.53; □, 0.74, ——, Calculation (Gaussian assumption) for $y/\delta = 0.08$.

The close agreement between the calculation and the measurements is not unexpected in view of the reasonable agreement between the measured and calculated R_{uv} in Fig. 5. Although the agreement for the R_{uv} results is poorer at large τ than at small τ, the discrepancy is less noticeable in the ϕ_{uv} results, probably because of the scatter over the low-frequency range of the measured spectrum.

Note that the spectrum ϕ_{uv} could, of course, have been obtained by taking the Fourier transform of Eq. (9). The result, involving the frequency convolution theorem, can be written as

$$\frac{\phi_{uv}(\omega)}{\langle u^2 \rangle \langle v^2 \rangle} = \int_0^\infty \phi_u(\omega_l)\phi_v(\omega - \omega_l)d\omega_l + \int_0^\infty \phi_{u,v}(\omega_l)\phi_{v,u}(\omega - \omega_l)d\omega_l \tag{14}$$

and ϕ_{uv} is derived from a knowledge of the spectra ϕ_u, ϕ_v and cross-spectra $\phi_{u,v}$, $\phi_{v,u}$ or a knowledge of the corresponding correlation functions. The second term on the right of (14) is real since it is the Fourier transform of the second term on the right of (9), which is an even function of τ.

Measured spectral densities of $u^2 - \langle u^2 \rangle$ and $v^2 - \langle v^2 \rangle$ are given in Figs. 8 and 9, respectively, normalized such that the area under the spectrum is equal to unity, i.e.

$$\int_0^\infty \phi_{u^2}\left(\frac{\omega\delta}{U_l}\right) d\left(\frac{\omega\delta}{U_l}\right) = \langle u^4 \rangle - \langle u^2 \rangle^2$$
$$\int_0^\infty \phi_{v^2}\left(\frac{\omega\delta}{U_l}\right) d\left(\frac{\omega\delta}{U_l}\right) = \langle v^4 \rangle - \langle v^2 \rangle^2 \tag{15}$$

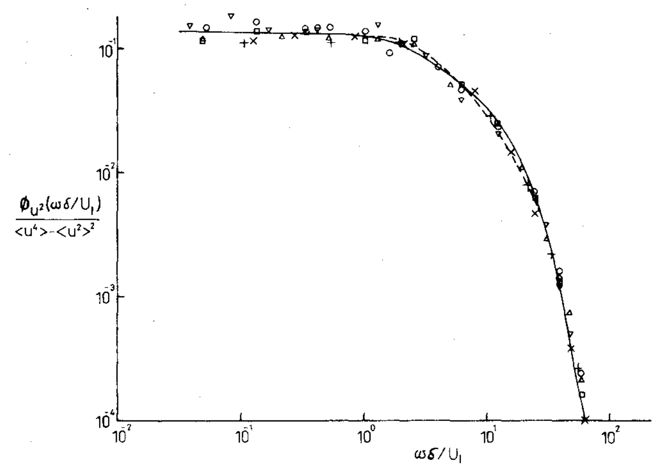

Fig. 8 Spectra of normal stress $u^2 - \langle u^2 \rangle$ fluctuations at several positions across the layer. Measurements: +, $y/\delta = 0.08$; Δ, 0.16; ○, 0.21; □, 0.53; ∇, 0.74; ×, 0.95. Calculations (Gaussian assumption): ——, $y/\delta = 0.08$; – –, 0.74.

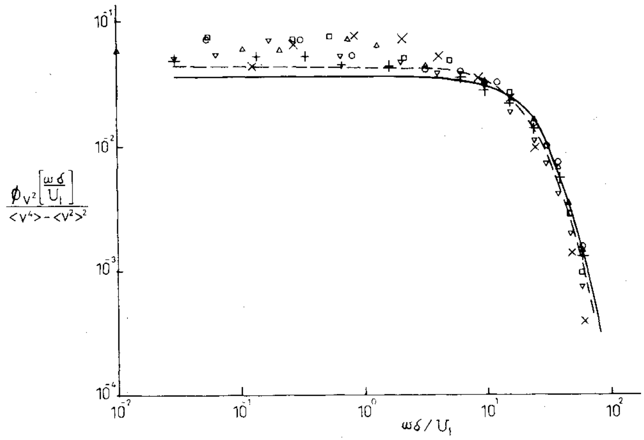

Fig. 9 Spectra of normal stress $v^2 - \langle v^2 \rangle$ fluctuations at several positions across the layer. Measurements: symbols as for Fig. 8. Calculations (Gaussian assumption); ——, $y/\delta = 0.08$; – –, 0.74.

These spectra are, like the ϕ_{uv} spectra, closely similar at all values of y/δ and over the whole frequency range considered here. The measured spectra are compared with theoretical calculations for these higher order spectra based on the assumption that $u(t)$ and $u(t+\tau)$ are jointly Gaussian, or that $v(t)$ and $v(t+\tau)$ are jointly Gaussian. General expressions for higher order spectra ϕ_{un} of arbitrary powers of a Gaussian variable $u(t)$ in terms of its correlation function $R(\tau)$ have been given by Rice[17] and by Van Atta and Wyngaard.[7] For present purposes, we need only the result for the square of a variable,

$$\phi_{u^2}(f) = \langle u^2 \rangle \left[\delta(f) + 2\int_{-\infty}^{\infty} R_u^2(\tau)e^{-i2\pi f\tau}d\tau\right] \tag{16}$$

An analogous formula gives $\phi_{v^2}(t)$ when the correlation function $R_v(\tau)$ of $v(t)$ is given.

The program used here was the same one employed by Van Atta and Wyngaard[7] to generate higher-order spectra up to fourth order for the mixing layer data of Champagne et al.[6] A three-section polynomial curve fit to the measured correlation function generated 4096 evenly spaced points of the correlation function. These data then were squared and fast-Fourier transformed as in Eq. (16). In all cases, the computed results for ϕ_u and ϕ_v are in close agreement with the measured first-order spectra, indicating that the results for ϕ_{u2} and ϕ_{v2} can be trusted.

The calculated ϕ_{u2} spectrum (Fig. 8) for $y/\delta=0.08$ is in close agreement with the measured spectrum. The calculated ϕ_{u2} at $y/\delta=0.74$ differs only slightly from that at $y/\delta=0.08$ and is surprisingly close to the data, in view of the definite intermittent nature of u at $y/\delta=0.74$. The calculated ϕ_{v2} spectrum (Fig. 9) for $y/\delta=0.08$ is in reasonable agreement with the measurements at high frequencies ($\omega\delta/U_I>10$) but underestimates the measured spectral densities at low frequencies. This latter result is not unexpected in view of the low values of the Gaussian correlation coefficient (Fig. 3) at large delay times. The departure from Gaussianity is more pronounced for the probability density of v than for that of u (e.g. Gupta and Kaplan[2]). The calculated spectrum of ϕ_{v2} at $y/\delta=0.74$ also underestimates the measured spectrum at low ω but, as for $y/\delta=0.08$, has a shape which is closely similar to that of the measured spectrum.

V. Conclusions

Autocorrelations and spectra of Reynolds stress fluctuations u^2, v^2 and uv have been measured in a turbulent boundary layer and compared with calculations obtained by assuming Gaussian individual and joint probability densities for the u and v fluctuations. In the case of u^2, the calculated correlations and spectrum are in good agreement with the measurements for all values of the delay time τ and the frequency, respectively. In the case of v^2 and uv, the agreement between the calculated and measured autocorrelation or spectrum is good only at low values of τ or high values of ω. For large values of τ or small values of ω, the calculated correlation coefficients and spectral densities are lower than the measured values. This discrepancy appears to be due to the fairly noticeable departure from Gaussianity of the probability density of the normal velocity fluctuations, probably as a result of the influence of the wall in inhibiting large-scale v fluctuations.

Acknowledgments

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